Euler's Rotation Theorem: *When a sphere is moved around its centre it is always possible to find a diameter whose direction in the displaced position is the same as in the initial position.*

To arrive at a proof, Euler analyses what the situation would look like if the theorem were true. To that end, suppose the yellow line goes through the center of the sphere and is the axis of rotation we are looking for, and point **O** is one of the two intersection points of that axis with the sphere. Then he considers an arbitrary great circle that does not contain **O** (the blue circle), and its image after rotation (the red circle), which is another great circle not containing **O.** He labels a point on their intersection as point **A**. (If the circles coincide, then **A** can be taken as any point on either; otherwise **A** is one of the two points of intersection.)

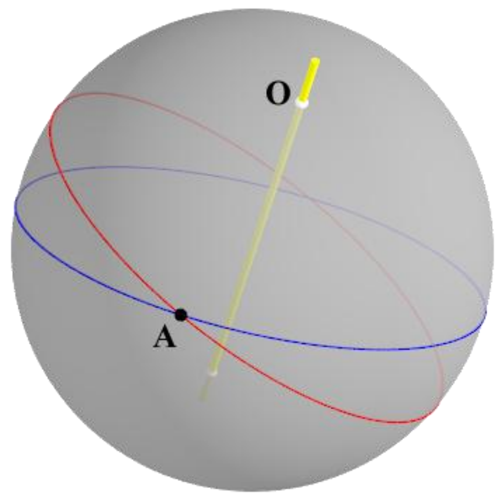

Figure 1: The blue circle is transformed into the red circle.

Now **A** is on the initial circle (the blue circle), so its image will be on the transported circle (red). He labels that image as point **a.** Since **A** is also on the transported circle (red), it is the image of another point that was on the initial circle (blue) and he labels that preimage as **α**. Then he considers the two arcs joining **α** and **a** to **A**. These arcs have the same length because arc **αA** is mapped onto arc **Aa**. Also, since **O** is a fixed point, triangle **αOA** is mapped onto triangle **AOa**, so these triangles are isosceles, and arc **AO** bisects angle **αAa**.

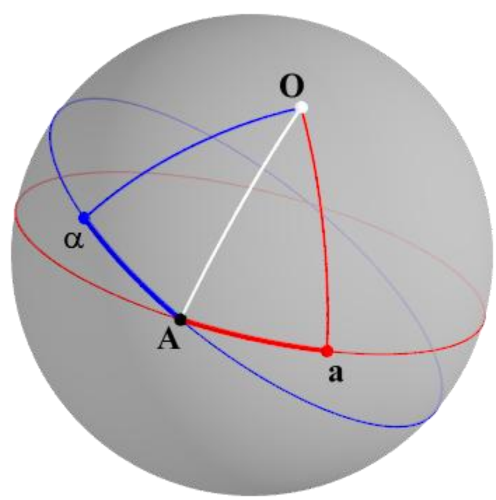

Figure 2: Arcs connecting preimage **α** and image **a** of **A** with bisector **AO** of angle **αAa**.

So here is the actual **proof**:

We start with the blue great circle and its image under the transformation, which is the red great circle as in Figure 1. Let point **A** be a point of intersection of those circles. If **A**'s image under the transformation is the same point then **A** is a fixed point of the transformation, and since the center is also a fixed point, the diameter of the sphere containing **A** is the axis of rotation and the theorem is proved.

Otherwise we label **A**'s image as **a** and its preimage as **α**, and connect these two points to **A** with arcs **αA** and **Aa**. These arcs have the same length. Construct the great circle that bisects angle **αAa** and locate point **O** on that great circle so that arcs **AO** and **aO** have the same length, and call the region of the sphere containing **O** and bounded by the blue and red great circles the "interior" of angle **αAa**. (That is the yellow region in Figure 3.) Then since **αA** = **Aa** and **O** is on the bisector of angle **αAa**, we also have **αO** = **aO**.

Now suppose **O'** is the image of **O**. Then we know angle **αAO** = angle **AaO'** and orientation is preserved*, so **O'** must be interior to angle **αAa**. Now **AO** is transformed to **aO'**, so **AO** = **aO'**. Since **AO** is also the same length as **aO**, angle **AaO** = angle **aAO**. But angle **aAO** = angle **AaO'**, so angle **AaO** = angle **AaO'** and therefore **O'** is the same point as **O**. In other words, **O** is a fixed point of the transformation, and since the center is also a fixed point, the diameter of the sphere containing **O** is the axis of rotation.

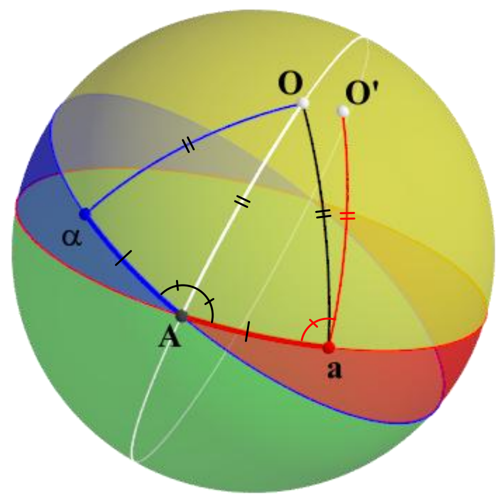

Figure 3: **O** goes to **O'**, but **O'** must coincide with **O**.

Euler also points out that **O** can be found by intersecting the perpendicular bisector of **Aa** with the angle bisector of angle **αAO**.

*Note: Orientation is preserved in the sense that if **αA** is rotated about **A** counterclockwise to align with **Oa**, then **Aa** must be rotated about **a** counterclockwise to align with **O'a**. Likewise if the rotations are clockwise.