Proof that a nonzero rational power of *e* is irrational (Joe Mercer, March 10, 2014)

Definition: A number is *irrational* iff no positive (integral) multiple of it is an integer.

Theorem: If p is a positive integer, then e^p is irrational.

Proof: Suppose m and n are positive integers, and let

$$I_n = 1/n! \int_0^\infty [x(x-p)]^n e^{-x} dx \quad \text{and} \quad J_n = 1/n! \int_0^\infty [x(x+p)]^n e^{-x} dx$$

Then I_n and J_n are integers because $(x \pm p)^n$ are polynomials with integer coefficients,

$[x(x \pm p)]^n = x^n (x \pm p)^n$, and $\int_0^\infty x^k e^{-x} dx = k!$ when k is a nonnegative integer.

Now consider the positive integral multiple of e^p, me^p, and multiply it by the integer I_n:

$$\begin{aligned} me^p\, I_n &= \frac{me^p}{n!}\int_0^p [x(x-p)]^n e^{-x} dx + \frac{m}{n!}\int_p^\infty [x(x-p)]^n e^{-(x-p)} dx \\ &= \frac{me^p}{n!}\int_0^p [x(x-p)]^n e^{-x} dx + \frac{m}{n!}\int_0^\infty [(u+p)u]^n e^{-u} du \qquad (\text{where } u = x-p) \\ &= \frac{me^p}{n!}\int_0^p [x(x-p)]^n e^{-x} dx + mJ_n \end{aligned}$$

On $[0,p]$: $|x(x-p)| \le \frac{p^2}{4}$ and $0 < e^{-x} \le 1$, so $\left|\frac{me^p}{n!}\int_0^p [x(x-p)]^n e^{-x} dx\right| \le \frac{me^p p^{2n}}{4^n n!}$

If n is chosen so that $n! > me^p \left(\frac{p^2}{4}\right)^n$ (possible since factorials grow faster than exponentials)

then $\left|\frac{me^p}{n!}\int_0^p [x(x-p)]^n e^{-x} dx\right| < 1$. Also, $\int_0^p [x(x-p)]^n e^{-x} dx \ne 0$ because

$[x(x-p)]^n e^{-x}$ does not change sign in $(0,\text{p})$. Thus $me^p I_n = \varepsilon_n + mJ_n$ where

$0 < |\varepsilon_n| < 1$ if $n! > me^p \left(\frac{p^2}{4}\right)^n$, and is not an integer in this case.

Therefore me^p must not be an integer. QED

Corollary: If p is a positive integer and q is a nonzero integer, then $e^{p/q}$ is irrational.

Proof: If $e^{p/q}$ were rational, then $\left(e^{p/q}\right)^q$ would be rational, contradicting the theorem.