Proof that *e* is not a quadratic number (Joe Mercer, July 23, 2013)

Definition: A number is *quadratic* iff it is the root of a quadratic equation with integer coefficients.

Proof: If e were a quadratic number, then there would be integers $m_1$and m_2 with $m_2 \neq 0$ such that $m_1 e + m_2 e^2$ is an integer.

Thus suppose $m_1$and m_2 are integers with $m_2 \neq 0$. Let *n* be an integer with certain conditions to be determined, and let

$$I_n = \int_0^\infty [x(x-1)(x-2)]^n e^{-x} dx, \qquad J_n = \int_0^\infty [x(x+1)(x-1)]^n e^{-x} dx,$$

$$K_n = \int_0^\infty [x(x+1)(x+2)]^n e^{-x} dx$$

Then I_n, J_n and K_n are integers because $[x(x-1)(x-2)]^n$, $[x(x+1)(x-1)]^n$ and $[x(x+1)(x+2)]^n$ are polynomials with integer coefficients, and $\int_0^\infty x^k e^{-x} dx = k!$ when k is a nonnegative integer.

Now consider $m_1 e + m_2 e^2$ and multiply it by the integer I_n:

$$(m_1 e + m_2 \mathrm{e}^2) I_n =$$

$$m_1 e \int_0^1 [x(x-1)(x-2)]^n e^{-x} dx \quad + \quad m_2 e^2 \int_0^2 [x(x-1)(x-2)]^n e^{-(x-1)} dx \quad +$$

$$m_1 \int_1^\infty [x(x-1)(x-2)]^n e^{-(x-1)} dx \; + \; m_2 \int_2^\infty [x(x-1)(x-2)]^n e^{-(x-2)} dx$$

$$= (m_1 e + m_2 e^2) A_n + m_2 e^2 B_n +$$

$$m_1 \int_0^\infty [(u+1)u(u-1)]^n e^{-u} du + m_2 \int_0^\infty [(v+2)(v+1)v]^n e^{v} dv$$

$$= (m_1 e + m_2 e^2) A_n + m_2 e^2 B_n + m_1 J_n + m_2 K_n$$

where $u = x - 1$ and $v = x - 2$, and A_n and B_n are as follows:

$$A_n = \int_0^1 [x(x-1)(x-2)]^n e^{-x} dx, \quad B_n = \int_1^2 [x(x-1)(x-2)]^n e^{-x} dx$$

Now on (0, 1), $x(x-1)(x-2)$ is positive, so A_n is positive. On the other hand, on (1, 2), $x(x-1)(x-2)$ is negative, so B_n would be negative if *n* is odd, but positive if *n* is even. Thus, if $m_1 e + m_2 e^2$ and $m_2 e^2$ have the same sign let *n* be even, otherwise let *n* be odd. Then $(m_1 e + m_2 e^2)A_n + m_2 e^2 B_n$ is nonzero. Let us label this as ε_n.

Furthermore, $|x(x-1)(x-2)| \leq \frac{2}{3\sqrt{3}} < 0.4$ on [0, 2], so $|A_n| < (0.4)^n$ and $|B_n| < (0.4)^n$, so

$0 < |\varepsilon_n| < |m_1 e + 2m_2 e^2|(0.4)^n$. Since $(0.4)^n \to 0$ as $n \to \infty$, we can choose *n* large enough so $|\varepsilon_n| < 1$.

Thus $(m_1 e + m_2 \mathrm{e}^2)I_n = \varepsilon_n + m_1 J_n + m_2 K_n$ where $m_1 J_n$ and $m_2 K_n$ are integers, but ε_n is nonzero and less than 1 in absolute value when *n* is chosen appropriately, so $(m_1 e + m_2 \mathrm{e}^2)I_n$ is not an integer, and therefore $m_1 e + m_2 \mathrm{e}^2$ is not an integer. QED

Test your understanding:

1. Prove by mathematical induction and integration by parts that $\int_0^\infty x^k e^{-x} dx = k!$ when k is a positive integer.

2. Prove that $|x(x-1)(x-2)| \leq \frac{2}{3\sqrt{3}}$ when $0 \leq x \leq 2$

3. Suppose we remove the restriction that *n* be even or odd depending on the signs of if $m_1 e + m_2 e^2$ and $m_2 e^2$. In that case, under what conditions might ε_n=0?