Proof that *e* is irrational (Joe Mercer, July 20, 2013)

Needed facts about *e* (what definition should we use? Perhaps $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$):

$$e > 1, \qquad e^{x+y} = e^x e^y, \qquad \frac{de^x}{dx} = e^x$$

Definition: A number is *irrational* iff no positive (integral) multiple of it is an integer.

Proof: Suppose *m* and *n* are positive integers, and let

$$I_n = \int_0^\infty [x(x-1)]^n e^{-x} dx \quad \text{and} \quad J_n = \int_0^\infty [x(x+1)]^n e^{-x} dx$$

Then I_n and J_n are integers because $[x(x \pm 1)]^n$ are polynomials with integer coefficients, and

$\int_0^\infty x^k e^{-x} dx = k!$ when k is a nonnegative integer.

Now consider the positive integral multiple of *e*, *me*, and multiply it by the integer I_n:

$$\begin{aligned} me\, I_n &= me \int_0^1 [x(x-1)]^n e^{-x} dx + m \int_1^\infty [x(x-1)]^n e^{-(x-1)} dx \\ &= me \int_0^1 [x(x-1)]^n e^{-x} dx + m \int_0^\infty [(u+1)u]^n e^{-u} du \qquad \text{(where } u = x-1\text{)} \\ &= me \int_0^1 [x(x-1)]^n e^{-x} dx + mJ_n \end{aligned}$$

On $[0,1]$: $|x(x-1)| \le \frac{1}{4}$ and $0 < e^{-x} \le 1$, so $\left| me \int_0^1 [x(x-1)]^n e^{-x} dx \right| \le \frac{me}{4^n}$

If $n > \log_4(me)$ then $\left| me \int_0^1 [x(x-1)]^n e^{-x} dx \right| < 1$

Also $\int_0^1 [x(x-1)]^n e^{-x} dx \ne 0$ because $[x(x-1)]^n e^{-x}$ does not change sign in $(0,1)$.

Thus $meI_n = \varepsilon_n + mJ_n$ where $0 < |\varepsilon_n| < 1$ if $n > \log_4(me)$, and is not an integer in this case .

Therefore me must not be an integer. QED